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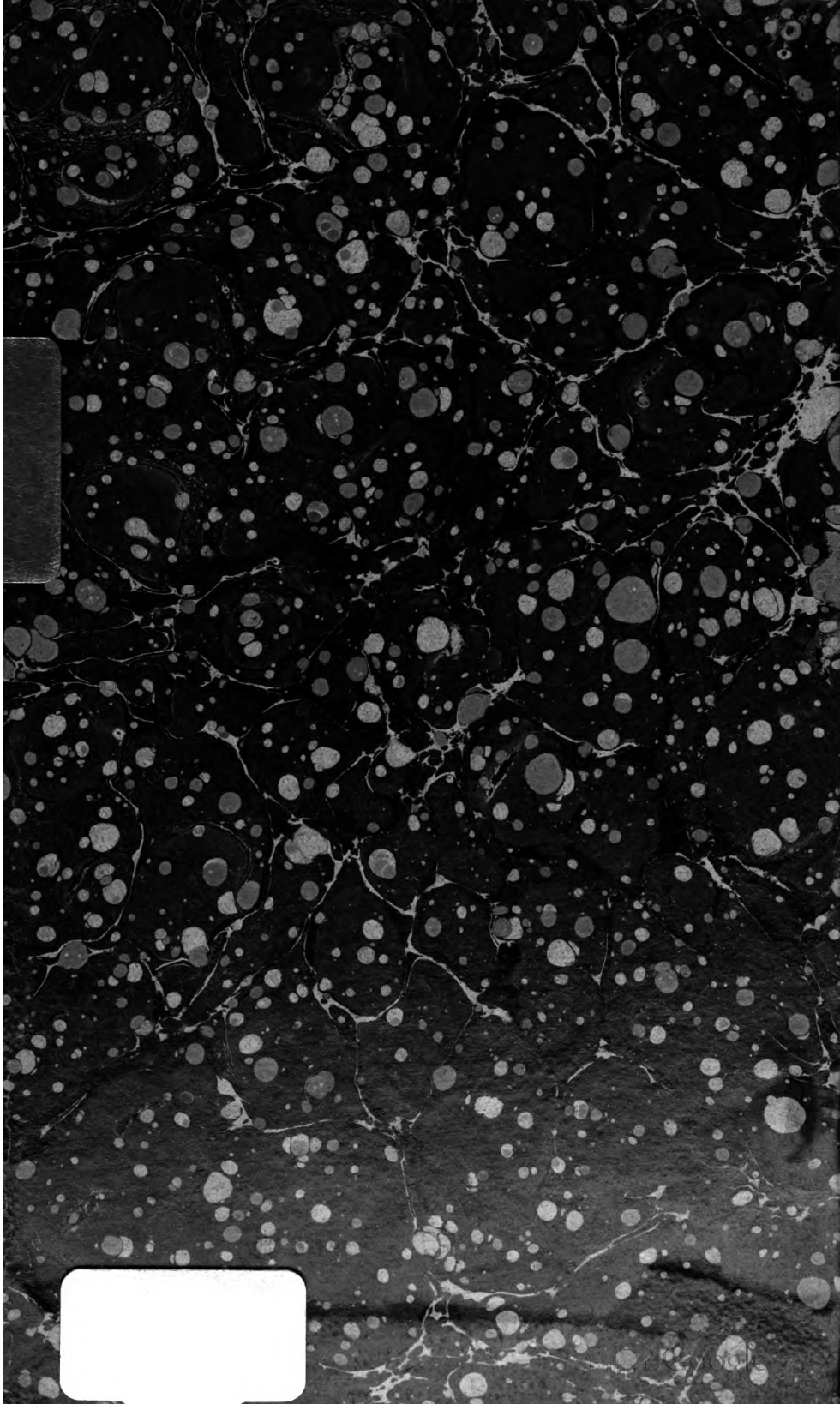
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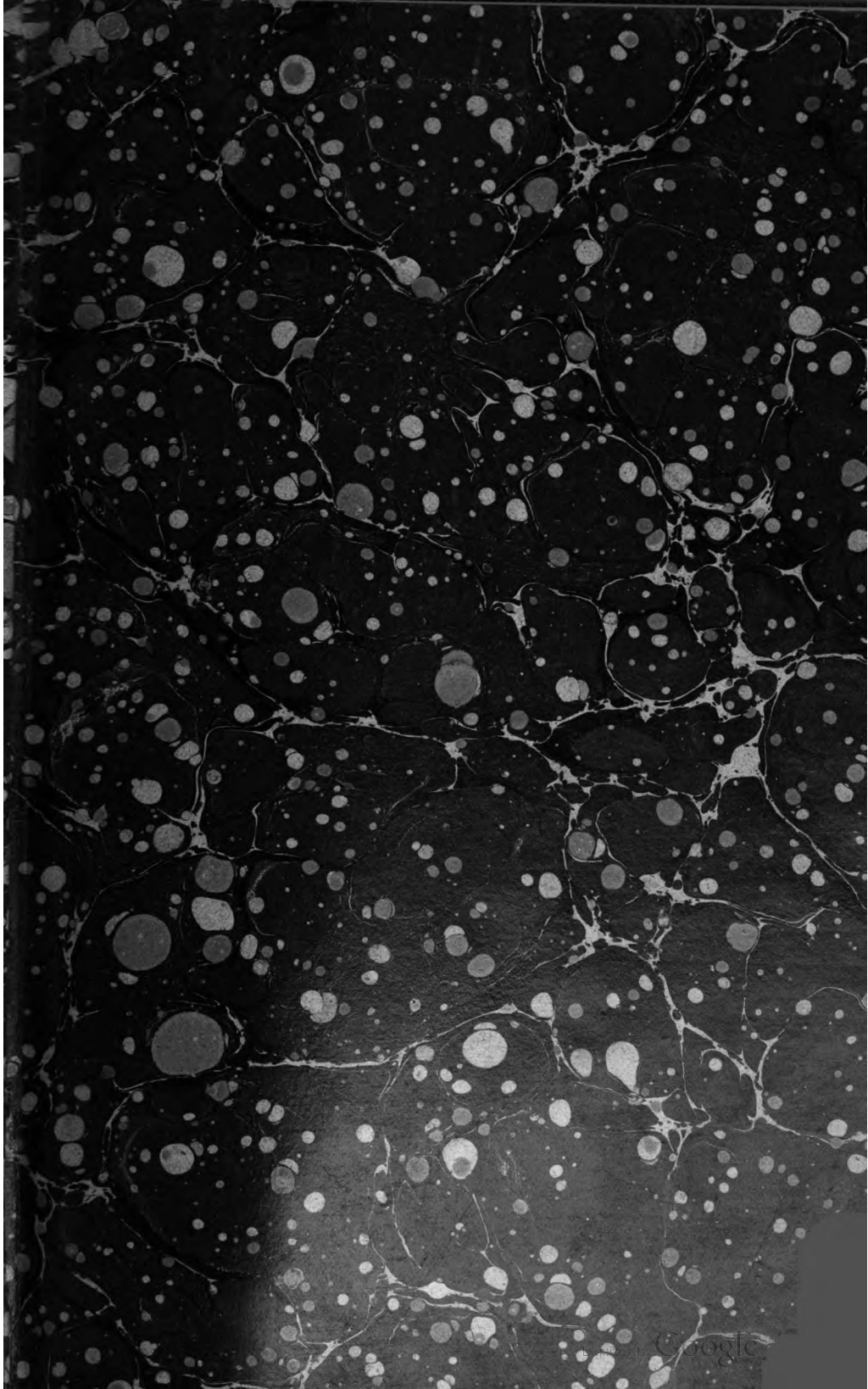
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ON A GENERALISATION OF IVORY'S THEOREM.

By A. L. Dixon, Merton College, Oxford.

IN Cayley's theory of distance (sixth memoir on quantics, *Coll. Works*, Vol. II., pp. 583–592), as applied to three dimensional space, a certain conicoid—suppose its equation to be $\phi(x, y, z, w) = 0$ —is called the “absolute,” and the “distance” s' between two points P_1, P_2 , whose coordinates are $(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2)$ is given by

$$s' = \cos^{-1} \frac{1}{2} \frac{x_1 \frac{\partial \phi}{\partial x_2} + y_1 \frac{\partial \phi}{\partial y_2} + z_1 \frac{\partial \phi}{\partial z_2} + w_1 \frac{\partial \phi}{\partial w_2}}{\sqrt{(\phi_1, \phi_2)}}$$

The ordinary formula for the distance occurs when the absolute is taken to be the conic in which the plane at infinity is cut by $x^2 + y^2 + z^2 = 0$, and in fact

$$s = \{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}^{\frac{1}{2}}$$

may be deduced from the preceeding formula, as a limiting case.

Ivory's theorem is as follows:—“If P and Q are two points on a conicoid, and P' and Q' are the points corresponding to them, respectively, on another conicoid which is confocal with the first, then the distance PQ' is equal to the distance $P'Q$,” where corresponding points are points which lie on the curve of intersection of the same two confocals of the given system.

I propose to show in this paper that *Ivory's theorem is true whenever the “absolute” is a conicoid of the confocal system.*

It is easy to see that the imaginary circle at infinity is a degenerate conicoid of any system of confocal conicoids, bearing the same relation to the system that the focal conics in the principal planes do, so that for the ordinary notion of distance the absolute is a member of any confocal system.

In the first section of this paper I prove this generalisation of Ivory's theorem, and also show that the lines joining two pairs of corresponding points both touch the same two confocals of the system. The proof is given for three dimensions, but is, of course, equally applicable to two, or any number of dimensions.

In the second section I apply the theorem to obtain a generalisation of the property that the sum of the focal distances of any point on a conic is constant, and in the third I consider the extension of Ivory's theorem to geodesic distance on an ellipsoid.

I.

§1. Taking

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1$$

as the equation of a set of confocal conicoids, I take

$$\frac{x^2}{a+n} + \frac{y^2}{b+n} + \frac{z^2}{c+n} = 1$$

as the "absolute."

Now if p, q, r are the values of λ for the three conicoids of the system through any point (x, y, z) , we have

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} - 1 \equiv -\frac{(\lambda-p)(\lambda-q)(\lambda-r)}{(a+\lambda)(b+\lambda)(c+\lambda)};$$

and therefore

$$(a-b)(a-c)x^2 = (a+p)(a+q)(a+r),$$

$$(b-c)(b-a)y^2 = (b+p)(b+q)(b+r),$$

$$(c-a)(c-b)z^2 = (c+p)(c+q)(c+r),$$

$$\frac{x^2}{a+n} + \frac{y^2}{b+n} + \frac{z^2}{c+n} - 1 = -\frac{(n-p)(n-q)(n-r)}{(a+n)(b+n)(c+n)}.$$

Then the distance between two points P_1, P_2 whose elliptic coordinates are $(p_1, q_1, r_1), (p_2, q_2, r_2)$, respectively, is given by

$$\begin{aligned} \cos \alpha &= \frac{\frac{x_1 x_2}{a+n} + \frac{y_1 y_2}{b+n} + \frac{z_1 z_2}{c+n} - 1}{\left(\frac{x_1^2}{a+n} + \frac{y_1^2}{b+n} + \frac{z_1^2}{c+n} - 1 \right)^{\frac{1}{2}} \left(\frac{x_2^2}{a+n} + \frac{y_2^2}{b+n} + \frac{z_2^2}{c+n} - 1 \right)^{\frac{1}{2}}} \\ &= \frac{(a+n)(b+n)(c+n)}{\sqrt{\{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)\}}} \\ &\times \left\{ \frac{\sqrt{\{(a+p_1)(a+q_1)(a+r_1)(a+p_2)(a+q_2)(a+r_2)\}}}{(a+n)(a-b)(a-c)} - 1 \right\}. \end{aligned}$$

Now the elliptic coordinates of the pair of points corresponding to P_1 and P_2 are (p_1, q_1, r_1) and (p_2, q_2, r_2) , and if we interchange p_1 and p_2 in the expression for $\cos \alpha$ it remains unaltered, so that Ivory's theorem is true when the absolute is a member of the confocal system.

§2. If we consider the tangent cone to

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1,$$

from any point (x', y', z') , we know that its equation is

$$\left(\Sigma \frac{x'^2}{a+\lambda} - 1\right) \left(\Sigma \frac{x'^2}{a+\lambda} - 1\right) = \left(\Sigma \frac{xx'}{a+\lambda} - 1\right)^2,$$

which is equivalent to

$$\frac{\xi_p^2}{\lambda-p'} + \frac{\xi_q^2}{\lambda-q'} + \frac{\xi_r^2}{\lambda-r'} = 0;$$

where ξ_p, ξ_q, ξ_r are the coordinates of any point, when the axes are the normals to the three confocals p', q', r' at the point (x', y', z') .

Using this result to obtain another expression for s' , I have

$$\begin{aligned} \sin^2 s' &= \frac{\left(\Sigma \frac{x_1^2}{a+n} - 1\right) \left(\Sigma \frac{x_2^2}{a+n} - 1\right) - \left(\Sigma \frac{x_1 x_2}{a+n} - 1\right)^2}{\left(\Sigma \frac{x_1^2}{a+n} - 1\right) \left(\Sigma \frac{x_2^2}{a+n} - 1\right)} \\ &= -\frac{(a+n)(b+n)(c+n)}{(n-p_2)(n-q_2)(n-r_2)} \left\{ \frac{\xi_p^2}{n-p_1} + \frac{\xi_q^2}{n-q_1} + \frac{\xi_r^2}{n-r_1} \right\}, \end{aligned}$$

where ξ_p, ξ_q, ξ_r are the coordinates of (x, y, z) , referred to the normals to the three confocals p, q, r at (x, y, z) .

I have also

$$s^2 = \xi_p^2 + \xi_q^2 + \xi_r^2.$$

Now suppose that the line joining (x_1, y_1, z_1) and (x, y, z) touches the two confocals whose parameters are λ, μ .

Then

$$\frac{\xi_p^2}{\lambda-p_1} + \frac{\xi_q^2}{\lambda-q_1} + \frac{\xi_r^2}{\lambda-r_1} = 0,$$

$$\frac{\xi_p^2}{\mu-p_1} + \frac{\xi_q^2}{\mu-q_1} + \frac{\xi_r^2}{\mu-r_1} = 0,$$

so that

$$\begin{aligned} \frac{\xi_p^2}{(\lambda-p_1)(\mu-p_1)(q_1-r_1)} &= \frac{\xi_q^2}{(\lambda-q_1)(\mu-q_1)(r_1-p_1)} \\ &= \frac{\xi_r^2}{(\lambda-r_1)(\mu-r_1)(p_1-q_1)} = \frac{-s^2}{(q_1-r_1)(r_1-p_1)(p_1-q_1)} \\ &= \sin^2 s' \frac{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)}{(a+n)(b+n)(c+n)(\lambda-n)(\mu-n)(q_1-r_1)(r_1-p_1)(p_1-q_1)}, \end{aligned}$$

and therefore

$$\sin^2 s' = -s^2 \frac{(a+n)(b+n)(c+n)(\lambda-n)(\mu-n)}{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)}.$$

I.

§1. Taking

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1$$

as the equation of a set of confocal conicoids, I take

$$\frac{x^2}{a+n} + \frac{y^2}{b+n} + \frac{z^2}{c+n} = 1$$

as the "absolute."

Now if p, q, r are the values of λ for the three conicoids of the system through any point (x, y, z) , we have.

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} - 1 \equiv - \frac{(\lambda-p)(\lambda-q)(\lambda-r)}{(a+\lambda)(b+\lambda)(c+\lambda)};$$

and therefore

$$(a-b)(a-c)x^2 = (a+p)(a+q)(a+r),$$

$$(b-c)(b-a)y^2 = (b+p)(b+q)(b+r),$$

$$(c-a)(c-b)z^2 = (c+p)(c+q)(c+r),$$

$$\frac{x^2}{a+n} + \frac{y^2}{b+n} + \frac{z^2}{c+n} - 1 = - \frac{(n-p)(n-q)(n-r)}{(a+n)(b+n)(c+n)}.$$

Then the distance between two points P_1, P_2 whose elliptic coordinates are $(p_1, q_1, r_1), (p_2, q_2, r_2)$, respectively, is given by

$$\begin{aligned} \cos s &= \frac{\frac{x_1 x_2}{a+n} + \frac{y_1 y_2}{b+n} + \frac{z_1 z_2}{c+n} - 1}{\left(\frac{x_1^2}{a+n} + \frac{y_1^2}{b+n} + \frac{z_1^2}{c+n} - 1 \right)^{\frac{1}{2}} \left(\frac{x_2^2}{a+n} + \frac{y_2^2}{b+n} + \frac{z_2^2}{c+n} - 1 \right)^{\frac{1}{2}}} \\ &= \frac{(a+n)(b+n)(c+n)}{\sqrt{\{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)\}}} \\ &\times \left\{ \Sigma \frac{\sqrt{\{(a+p_1)(a+q_1)(a+r_1)(a+p_2)(a+q_2)(a+r_2)\}}}{(a+n)(b+n)(c+n)} - 1 \right\}. \end{aligned}$$

Now the elliptic coordinates of the pair of points corresponding to P_1 and P_2 are (p_1, q_1, r_1) and (p_2, q_2, r_2) . If we interchange p_1, q_1, r_1 with p_2, q_2, r_2 , the value of the expression remains unaltered. This is a consequence of Ivory's theorem. The absolute is a meridian of the conicoids.

§2. If we c

from any point (x', y', z') , we know that its equation is

$$\left(\Sigma \frac{x'^2}{a+\lambda} - 1\right) \left(\Sigma \frac{x'^2}{a+\lambda} - 1\right) = \left(\Sigma \frac{xx'}{a+\lambda} - 1\right)^2,$$

which is equivalent to

$$\frac{\xi_p^2}{\lambda-p'} + \frac{\xi_q^2}{\lambda-q'} + \frac{\xi_r^2}{\lambda-r'} = 0;$$

where ξ_p, ξ_q, ξ_r are the coordinates of any point, when the axes are the normals to the three confocals p', q', r' at the point (x', y', z') .

Using this result to obtain another expression for s' , I have

$$\begin{aligned} \sin^2 s' &= \frac{\left(\Sigma \frac{x_1^2}{a+n} - 1\right) \left(\Sigma \frac{x_2^2}{a+n} - 1\right) - \left(\Sigma \frac{x_1 x_2}{a+n} - 1\right)^2}{\left(\Sigma \frac{x_1^2}{a+n} - 1\right) \left(\Sigma \frac{x_2^2}{a+n} - 1\right)} \\ &= -\frac{(a+n)(b+n)(c+n)}{(n-p_1)(n-q_1)(n-r_1)} \left\{ \frac{\xi_p^2}{n-p_1} + \frac{\xi_q^2}{n-q_1} + \frac{\xi_r^2}{n-r_1} \right\}, \end{aligned}$$

where ξ_p, ξ_q, ξ_r are the coordinates of (x_1, y_1, z_1) , referred to the normals to the three confocals p_1, q_1, r_1 at (x_1, y_1, z_1) .

I have also

$$s^2 = \xi_p^2 + \xi_q^2 + \xi_r^2.$$

Now suppose that the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) touches the two confocals whose parameters are λ, μ .

Then

$$\frac{\xi_p^2}{\lambda-p_1} + \frac{\xi_q^2}{\lambda-q_1} + \frac{\xi_r^2}{\lambda-r_1} = 0,$$

$$\frac{\xi_p^2}{\mu-p_1} + \frac{\xi_q^2}{\mu-q_1} + \frac{\xi_r^2}{\mu-r_1} = 0,$$

so that

$$\begin{aligned} \frac{\xi_p^2}{(\lambda-p_1)(\mu-p_1)(q_1-r_1)} &= \frac{\xi_q^2}{(\lambda-q_1)(\mu-q_1)(r_1-p_1)} \\ &= \frac{\xi_r^2}{(\lambda-r_1)(\mu-r_1)(p_1-q_1)} = \frac{-s^2}{(q_1-r_1)(r_1-p_1)(p_1-q_1)} \\ &= \sin^2 s' \frac{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)}{(a+n)(b+n)(c+n)(\lambda-n)(\mu-n)(q_1-r_1)(r_1-p_1)(p_1-q_1)}, \end{aligned}$$

and therefore

$$\sin^2 s' = -s^2 \frac{(a+n)(b+n)(c+n)(\lambda-n)(\mu-n)}{(n-p_1)(n-q_1)(n-r_1)(n-p_2)(n-q_2)(n-r_2)}.$$

§3. Now as both s and s' are unaltered, when for the two points P_1 and P_2 are substituted the points corresponding to them, we see that $(\lambda - n)(\mu - n)$ is also unaltered. This is true for any value of n , and therefore λ and μ are the same for pairs of corresponding points; and we get the proposition that the straight lines joining pairs of corresponding points touch the same two conicoids of the confocal system.

II.

§4. A generalisation of the property that the sum of the focal distances of any point on a conic is constant, is an interesting deduction from the result of §1. It is convenient to change the notation a little so as to distinguish the four points of intersection of two conics between themselves.

Taking

$$x = c \cos u \cos v,$$

$$y = ic \sin u \sin v,$$

the four points of intersection of the two conics whose parameters are u and v , considered as lying on the conic $u = \text{const.}$, are given by the coordinates

$$(u, v), (u, \pi - v), (u, \pi + v), (u, -v).$$

The line joining the real foci, when considered as a conic of the system, has the parameter 0, or π , and the coordinates of the two foci, S, H , may be taken to be $(0, 0)$ and $(0, \pi)$ respectively.

Then $SP \equiv$ distance between $(0, 0)$ and (u, v)

$$= \text{distance between } (u, 0) \text{ and } (0, v),$$

and $HP \equiv$ distance between $(0, \pi)$ and (u, v)

$$= \text{distance between } (u, \pi) \text{ and } (0, v),$$

and therefore $SP \pm HP = \text{const.}$ when u is constant, being equal to the distance between $(u, 0)$ and (u, π) , and this is true whenever the absolute is a member of the confocal system.

§5. Now, when the absolute is a proper conic, the distance between two points projects into the distance between two points, and therefore, by the method of projection, I obtain the following proposition:—

Taking Ω any given conic as the absolute, S , any other conic, has three pairs of foci, viz. the pairs of intersections of

the common tangents of S and Ω , and the sum or difference of the distances of any point on S from a pair of foci is constant.

§6. If we make use of Klein's expression for the distance between two points, viz.

$$\text{distance } PQ = c \log \frac{PA_1 \cdot QA_2}{PA_2 \cdot QA_1}, *$$

where A_1, A_2 are the points in which the line PQ meets the absolute, this last form of the proposition is seen to be equivalent to the following theorem in Euclidean geometry.

Given any two conics S, Ω and a pair of intersections of their common tangents A, B ; if the line PA joining any point P on S to A meet Ω in M_1, M_2 , and PB meet Ω in N_1, N_2 , then

$$\frac{AM_1 \cdot PM_2 \cdot BN_1 \cdot PN_2}{AM_2 \cdot PM_1 \cdot BN_2 \cdot PN_1},$$

or $\{AP, M_1M_2\} \{BP, N_1N_2\}$,

is constant for all positions of P .†

§7. In the particular case when Ω degenerates into the line joining two of the foci, that is to say into one of the diagonals of the circumscribed quadrilateral, M_1 coincides with M_2 , and N_1 with N_2 , and the expression for the distance AP becomes

$$\begin{aligned} \text{Lt. log} \left(1 + \frac{M_2M_1}{AM_1} \right) - \text{Lt. log} \left(1 + \frac{M_2M_1}{PM_1} \right) \\ = M_2M_1 \left(\frac{1}{AM_1} - \frac{1}{PM_1} \right) = \frac{M_2M_1 \cdot PA}{AM_1 \cdot PM_1}, \end{aligned}$$

and we get

$$\frac{(\text{Lt. } M_2M_1) PA}{AM_1 \cdot PM_1} + \frac{(\text{Lt. } N_2N_1) PB}{BN_1 \cdot PN_1} = \text{const.}$$

* PA_1, QA_2 , &c. in this expression means distances in the ordinary sense, c is a constant. The formula is, of course, the equivalent of Cayley's $\cos^2$ formula.

† As a verification, if we suppose that the line joining A and B meets S in P and Q and Ω in X and Y , we have that

$$\frac{PX^2 \cdot AY \cdot BY}{PY^2 \cdot AX \cdot BX} = \frac{QY^2 \cdot AX \cdot BX}{QX^2 \cdot AY \cdot BY},$$

or

$$\{APYX\}^2 = \{BQXY\}^2,$$

which is true since AB, PQ, XY are pairs in an involution, whose double points are the pair of points on AB which are conjugate to both S and Ω .

In the particular case of confocal conics, with Ω as the line joining the two circular points I, J at infinity, we have

$$\text{Lt. } M_1 M_1 = \text{Lt. } N_1 N_1,$$

for all directions, but in general this is not the case.

§8. For example, let us take as absolute the line joining the two imaginary foci of any ellipse, that is the minor axis. Then if C is the centre of the conic and S and H the foci, we shall find by considering Ω as the limit of

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

when $\lambda = -a^2$, that $\text{Lt. } M_1 M_1$ is proportional to $\sec^2 PSC$ and we shall get the proposition that

$$\frac{SP \sec^2 PSC}{SM \cdot PM} + \frac{HP \sec^2 PHC}{HN \cdot PN}$$

is constant, which is verified immediately, for it is equal to

$$\frac{SP}{SC \cdot QC} + \frac{HP}{HC \cdot QC}$$

(where Q is the foot of the perpendicular from P on the major axis), that is to

$$\frac{SP - HP}{SC \cdot QC} \text{ or } \frac{2}{a}.$$

If we take as absolute the line joining the real foci, we get an interesting expression for the distance of any point P from a focus.

In the general formula

$$\cos s' = \frac{\frac{x_1 x_2}{a+n} + \frac{y_1 y_2}{b+n} - 1}{\left(\frac{x_1^2}{a+n} + \frac{y_1^2}{b+n} - 1 \right)^{\frac{1}{2}} \left(\frac{x_2^2}{a+n} + \frac{y_2^2}{b+n} - 1 \right)^{\frac{1}{2}}}$$

put $y_2 = 0$, $x_2 = \sqrt{(a-b)}$, so that (x_2, y_2) is the focus S , and also put $n = -b + \theta$. Then

$$\cos s' = \frac{\frac{x_1 \sqrt{(a-b)}}{a-b+\theta} - 1}{\left[\frac{x_1^2}{a-b+\theta} + \frac{y_1^2}{\theta} - 1 \right]^{\frac{1}{2}} \left[\frac{a-b}{a-b+\theta} - 1 \right]^{\frac{1}{2}}},$$

and in the limit when $\theta = 0$ we shall get

$$\cos s' = \frac{x_1 - \sqrt{(a-b)}}{iy_1} = i \cot PSH,$$

or
$$\frac{\pi}{2} - s' = -i \log \tan \frac{1}{2} PSH.$$

From this we see that the well known property that for any point P on an ellipse $\tan \frac{1}{2} PSH \cdot \tan \frac{1}{2} PHS$ is constant may be considered a special case of the general theorem that the sum of the focal distances is constant, the absolute in this case being the line SH considered as a degenerate conic of the confocal system.

§9. The following particular cases of the general theorem seem worthy of notice.

(i) Two parabolas S, S' have a common axis and a common focus O . If P is any point on S , and if OP meet S' in X , PO meet it in Y , and a line through P parallel to the axis meet it in Q , then the ratio

$$PQ : \frac{OY \cdot PX}{OX \cdot PY}$$

is constant.

(a) Two conics are inscribed in a triangle ABC and touch BC in the same point D . P is a point on one conic and DP meets the other conic in M , AP meets it in M_1 , BP in Y_1 , CP in Z_1 , and PN is the perpendicular from P on BC .

Then

$$(i) \frac{BY_1 \cdot PY_1}{BY_2 \cdot PY_2} \cdot \frac{CZ_1 \cdot PZ_1}{CZ_2 \cdot PZ_2} \text{ is constant,}$$

and (ii) the ratio

$$PN : \frac{PM \cdot PM_1 \cdot AM_1}{DM \cdot PM_2 \cdot AM_2} \text{ is constant.}^*$$

* That the limit of PM' / DM' when M' coincides with D is proportional to the perpendicular from P on the tangent at D , can be seen as follows:—

Take $y = ax^2 + 2hxy + by^2$, as the equation to the conic and consider the ratios of the segments into which the line joining the points (ξ, η) and $(\theta, 0)$, is divided by the conic.

Putting

$$x = \frac{\xi + \lambda\theta}{1 + \lambda}, \quad y = \frac{\eta}{1 + \lambda},$$

these ratios are the roots of the equation

$$\eta(1 + \lambda) = a(\xi + \lambda\theta)^2 + 2h\eta(\xi + \lambda\theta) + b\eta^2,$$

and when θ is very small, one value of λ is approximately $\eta / a\theta^2$.

(b) Two conics have double contact at B and C , and their common tangents meet in A . P is a point on one conic, and AP meets the other conic in X , X_1 , BP meets it in Y , and CP in Z , and PM and PN are the perpendiculars on AB and AC respectively, then

$$(i) \frac{AX_1 \cdot PX_1}{AX \cdot PX} \text{ is constant,}$$

and

$$(ii) PM \cdot PN \cdot \frac{BY \cdot CZ}{PY \cdot PZ} \text{ is constant.}$$

If PL be the perpendicular from P on BC , the last result may be stated thus; PL is a constant multiple of

$$\frac{PY \cdot PZ}{BY \cdot CZ}.$$

§10. The correlative theorems derived from the foregoing by reciprocation are:—

If a, b are a pair of common chords of two conics, and p any tangent to one of them, and if m_1, m_2 are the tangents from (ap) to the other conic and n_1, n_2 the tangents from (bp) to it, then

$$\{ap, m_1 m_2\} \{bp, n_1 n_2\}$$

is constant.

(a) Two conics are described about a triangle abc , and have the same tangent d at the vertex (bc) . If p is any tangent to one of them and if m_1, m_2 are the tangents to the other from (ap) , and m is the second tangent from (pd) to it, then the perpendicular from (bc) on p is in constant ratio to

$$\frac{\sin(pm)}{\sin(dm)} \{ap, m_1 m_2\}.$$

(b) Two conics have double contact, b and c are the common tangents, and a the chord of contact. If p is any tangent to one of them and if m_1, m_2 are the tangents to the other from (ap) , and y and z the second tangents from (bp) and (cp) respectively, then (i) $\{ap, m_1 m_2\}$ is constant, and (ii) the product of the perpendiculars from (ab) and (ac) on to p , or the square of the perpendicular from (bc) on to p , is in constant ratio to

$$\frac{\sin(py) \cdot \sin(pz)}{\sin(by) \cdot \sin(cz)}.$$

III.

§ 11. In considering geodesics I take $r=0$ as the conicoid on which the geodesics are drawn, then p, q are the elliptic coordinates of a point on it.

The tangent line at any point on a geodesic touches a fixed conicoid, which I take to be the conicoid $r = \kappa$.

Then at any point (p, q) the equations of the tangent lines to the two geodesics drawn from that point to touch the conicoid $r = \kappa$ are

$$\xi_r = 0, \quad \frac{\xi_p^2}{\kappa - p} + \frac{\xi_q^2}{\kappa - q} = 0.$$

Therefore along a geodesic we have always

$$\frac{d\xi_p}{\sqrt{(p - \kappa)}} = \pm \frac{d\xi_q}{\sqrt{(\kappa - q)}}.$$

But $d\xi_p$ is an element of length along the direction in which q is constant, and moreover

$$\begin{aligned} \left(\frac{d\xi_p}{dp}\right)^2 &= \Sigma \left(\frac{dx}{dp}\right)^2 = \frac{1}{4} \Sigma \frac{x^2}{(a+p)^2} \\ &= \frac{1}{4} \frac{(p-q)p}{(a+p)(b+p)(c+p)}. \end{aligned}$$

Therefore the differential equation to a geodesic is

$$\frac{\sqrt{(p)} dp}{\sqrt{\{(a+p)(b+p)(c+p)(p-\kappa)\}}} = \pm \frac{\sqrt{(q)} dq}{\sqrt{\{(a+q)(b+q)(c+q)(q-\kappa)\}}}.$$

Also if s be the length of the arc of the geodesic,

$$ds^2 = d\xi_p^2 + d\xi_q^2,$$

and therefore

$$\begin{aligned} \frac{ds}{\sqrt{(p-q)}} &= \frac{d\xi_p}{\sqrt{(p-\kappa)}} = \pm \frac{d\xi_q}{\sqrt{(\kappa-q)}} \\ &= \pm \frac{\sqrt{\{(p-q)p\}} dp}{\sqrt{\{(a+p)(b+p)(c+p)(p-\kappa)\}}} = \pm \frac{\sqrt{\{(p-q)q\}} dq}{\sqrt{\{(a+q)(b+q)(c+q)(q-\kappa)\}}}, \\ ds &= \pm \frac{p^{\frac{1}{2}} dp}{\sqrt{\{p(a+p)(b+p)(c+p)(p-\kappa)\}}} \mp \frac{q^{\frac{1}{2}} dq}{\sqrt{\{q(a+q)(b+q)(c+q)(q-\kappa)\}}}, \end{aligned}$$

the upper or the lower signs being always taken together throughout this system of equations.

§ 12. Now let the geodesic through the two points (p_1, q_1) and (p_2, q_2) touch the conicoid $r = \kappa$, and suppose its equation to be

$$\frac{\sqrt{(p)} dp}{\sqrt{\{(a+p)(b+p)(c+p)(p-\kappa)\}}} - \frac{\sqrt{(q)} dq}{\sqrt{\{(a+q)(b+q)(c+q)(q-\kappa)\}}} = 0.$$

Then writing, for shortness, P for the product

$$p(a+p)(b+p)(c+p)(p-\kappa),$$

I have

$$\int_{p_1}^{p_2} \frac{p \, dp}{\sqrt{(P)}} - \int_{q_1}^{q_2} \frac{q \, dq}{\sqrt{(Q)}} = 0,$$

and the geodesic distance s between the points is given by

$$2s = \int_{p_1}^{p_2} \frac{p^2 \, dp}{\sqrt{(P)}} - \int_{q_1}^{q_2} \frac{q^2 \, dq}{\sqrt{(Q)}}.$$

But these two equations may be written

$$\int_{p_1}^{p_2} \frac{p \, dp}{\sqrt{(P)}} + \int_{q_2}^{q_1} \frac{q \, dq}{\sqrt{(Q)}} = 0,$$

$$2s = \int_{p_1}^{p_2} \frac{p^2 \, dp}{\sqrt{(P)}} + \int_{q_2}^{q_1} \frac{q^2 \, dq}{\sqrt{(Q)}},$$

and in this form the first expresses the condition that the points (p_1, q_2) and (p_2, q_1) lie on a geodesic whose differential equation is

$$\frac{p \, dp}{\sqrt{(P)}} + \frac{q \, dq}{\sqrt{(Q)}} = 0,$$

and the second gives the distance between the points (p_1, q_2) and (p_2, q_1) measured along that geodesic.

From which we see that the geodesics joining two pairs of corresponding points touch the same line of curvature and the geodesic distances between two pairs of corresponding points are equal.

§ 13. It is easy to see that this proposition also is true, whenever the absolute is a member of the confocal system. For, in the first place, since the normal and the tangent plane at any point are conjugate with respect to any conicoid of the confocal system, the geodesic remains the same curve, and secondly, we have for the element of length along it

$$ds' = - \frac{(a+n)(b+n)(c+n)}{n(n-p)(n-q)} \left\{ \frac{d\xi_p^2}{n-p} + \frac{d\xi_q^2}{n-q} \right\},$$

from which

$$\frac{2nids'}{\sqrt{\{n(a+n)(b+n)(c+n)(n-\kappa)\}}} = \frac{p \, dp}{(n-p)\sqrt{(P)}} \pm \frac{q \, dq}{(n-q)\sqrt{(Q)}},$$

and the same results will follow as before.

§ 14. So long as the absolute is a member of the confocal system the umbilics on the ellipsoid correspond to the foci of a plane confocal system, for the generators through an umbilic touch every line of curvature on the ellipsoid, so that the umbilics retain their characteristic properties of being points, the tangent plane at which meets the ellipsoid in a point circle.

The elliptic coordinates of the umbilics are $(-a, -a)$, $(-b, -b)$, and $(-c, -c)$, each pair corresponding to a set of four umbilics, and it is an immediate consequence of the foregoing that the sum or difference of the geodesic distances of any point on a line of curvature from two umbilics of the same set is constant.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XII.

On the operation which is the inverse of double integration.

§ 1. If $\phi(x, y)$ is a continuous function of x and y , and $a < x$, $b < y$,

$$\begin{aligned} \int_a^x dx \int_b^y \phi(x, y) dy &= \int_b^y dy \int_a^x \phi(x, y) dx \\ &= \int_a^x \int_b^y \phi(x, y) dx dy; \end{aligned}$$

and, if each of these functions is denoted by $f(x, y)$,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \phi(x, y).$$

The operations

$$\frac{\partial^2}{\partial x \partial y}, \quad \frac{\partial^2}{\partial y \partial x}$$

are evidently the inverses of

$$\int_b^y dy \int_a^x dx, \quad \int_a^x dx \int_b^y dy,$$

and it is of some theoretical interest to define the inverse of

$$\int^x \int^y dx dy.$$

Now

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk},$$

Let

$$D_{x,y} f = \lim_{h, k \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk}.$$

Then it is clear that

$$D_{x,y}$$

is the operation required. I shall call $D_{x,y} f$ (or simply Df) the *double differential coefficient* of $f(x, y)$. In the case in which $f(x, y)$ is the integral (double or repeated) of a continuous function, $D_{x,y} f$ is certainly determinate and equal to either of the repeated differential coefficients. But in general this is of course not the case. If, for instance,

$$\begin{aligned} f &= xy \left\{ 1 + \left(\frac{x-y}{x+y} \right)^2 \right\} \\ &= \frac{2xy(x^2 + y^2)}{(x+y)^2} \quad (x, y > 0) \end{aligned}$$

and

$$= 0 \quad (x \text{ or } y = 0),$$

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{x=0, y=0} &= \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \left[1 + \left(\frac{h-k}{h+k} \right)^2 \right] \\ &= 2, \end{aligned}$$

$$\left(\frac{\partial^2 f}{\partial y \partial x} \right)_{x=0, y=0} = 2,$$

but Df is indeterminate for $x=0, y=0$.

On the other hand, if

$$f(x, y) = \psi(x) + \chi(y),$$

$$Df \equiv 0,$$

whatever be the nature of the functions ψ, χ , provided only they are defined for all values of x and y respectively. We may, for instance, suppose ψ and χ never to have differential coefficients, or to be totally discontinuous.

§ 2. I shall now prove the following theorem :

If $f(A, B) - f(a, B) - f(A, b) + f(a, b) = 0$, and $Df(x, y)$ is continuous throughout the rectangle (a, A, b, B) , there is a point within the rectangle for which

$$Df(x, y) = 0.$$

I suppose $a < A, b < B$, and for brevity I shall write

$$[\alpha, \beta; \alpha', \beta']$$

for $f(\alpha, \beta) - f(\alpha', \beta) - f(\alpha, \beta') + f(\alpha', \beta')$.

If there is no such point within the rectangle, $Df(x, y)$ must always have the same sign, which we may suppose positive.

We divide the rectangle into $\mu\nu$ parts by drawing the lines

$$x = x_r^1 = a + \frac{r}{\mu} (A - a),$$

$$y = y_s^1 = b + \frac{s}{\nu} (B - b),$$

$$(r = 1, 2, \dots, \mu - 1; s = 1, 2, \dots, \nu - 1).$$

Then $0 = [A, B; a, b]$

$$= \sum_{r=0}^{\mu-1} \sum_{s=0}^{\nu-1} [x_{r+1}^1, y_{s+1}^1; x_r^1, y_s^1].$$

Therefore one at least of the terms $[x_{r+1}^1, y_{s+1}^1; x_r^1, y_s^1]$ is negative or zero. Suppose that the term for which $r = r_1, s = s_1$ is such a term.

We next divide the rectangle $(x_{r_1}^1, x_{r_1+1}^1, y_{s_1}^1, y_{s_1+1}^1)$ or R_1 into $\mu\nu$ parts in the same way, and so define a rectangle $(x_{r_2}^2, x_{r_2+1}^2, y_{s_2}^2, y_{s_2+1}^2)$ or R_2 , such that

$$[x_{r_2+1}^2, y_{s_2+1}^2; x_{r_2}^2, y_{s_2}^2] \leq 0;$$

and so on indefinitely.

It is evident that the rectangles $R_1, R_2, \dots$ define a limiting point (ξ, η) or P . If P is the lower left-hand corner of one of the rectangles R , which will be the case if, and only if, there is a number N such that

$$r_n = 0, s_n = 0 \quad (n > N),$$

it is plain that

$$Df(\xi, \eta) \leq 0.$$

$$\text{For } \frac{[x_1^n, y_1^n; \xi, \eta]}{(x_1^n - \xi)(y_1^n - \eta)} \leq 0 \quad (n > N),$$

and

$$x_1^n - \xi = \frac{A - a}{\mu^n},$$

$$y_1^n - \eta = \frac{B - b}{\nu^n},$$

which may be made as small as we please by choice of n . These equations involve $Df(\xi, \eta) \leq 0$.

A similar argument applies if P is one of the other corners of a rectangle R .

If P lies on a side (but not at a corner) of a rectangle R , or lies within all the rectangles, the proof that $Df(\xi, \eta) \leq 0$ is slightly more complicated. I choose the last case as the most difficult.

Divide each rectangle R_n into four parts by drawing the lines $x = \xi, y = \eta$. Then, since

$$\begin{aligned} & [x_{r_n+1}^n, y_{s_n+1}^n; x_{r_n}^n, y_{s_n}^n] \\ &= [\xi, \eta; x_{r_n}^n, y_{s_n}^n] + [x_{r_n+1}^n, \eta; \xi, y_{s_n}^n] \\ &+ [\xi, y_{s_n+1}^n; x_{r_n}^n, \eta] + [x_{r_n+1}^n, y_{s_n+1}^n; \xi, \eta], \end{aligned}$$

one at least of the four terms on the right is negative or zero. This applies to all the rectangles R_n . It does not follow that the *same* one of the four terms ≤ 0 for all values of n from a fixed value; but it does follow that *one at least* of the four ≤ 0 for an infinite series of values of n .

Suppose, for instance, that this is the case with the second term. Then

$$\frac{[x_{r_n+1}^n, \eta; \xi, y_{s_n}^n]}{(x_{r_n+1}^n - \xi)(\eta - y_{s_n}^n)} \leq 0,$$

$$\text{i.e. } \frac{[\xi, \eta; x_{r_n+1}^n, y_{s_n}^n]}{(\xi - x_{r_n+1}^n)(\eta - y_{s_n}^n)} \leq 0,$$

for an infinite series of values of n . And this involves

$$Df(\xi, \eta) \leq 0.$$

Thus in any case the assumption that Df is always positive leads to a contradiction, and the theorem follows.

§ 3. It might be thought that it would be simpler to introduce the function

$$F(x, y) = \int_a^x \int_b^y Df dx dy.$$

It is evident that if

$$(1) \quad F(A, B) - F(A, b) - F(a, B) + f(a, b) = 0,$$

there is a point within the rectangle for which

$$DF = Df = 0.$$

But we cannot deduce (1) from the conditions of the theorem without assuming that if

$$\phi = f - F,$$

so that

$$D\phi = 0,$$

then $\phi(A, B) - \phi(A, b) - \phi(a, B) + \phi(a, b) = 0$.

This is true, but to prove it we must assume the theorem of § 2.*

§ 4. *Theorem of the Mean.* We can now prove that if $Df(x, y)$ is continuous throughout (a, A, b, B) ,

$$\begin{aligned} f(A, B) - f(A, b) - f(a, B) + f(a, b) \\ = (A - a)(B - b) Df(\xi, \eta) \\ (a \leq \xi \leq A, b \leq \eta \leq B). \end{aligned}$$

For, consider the function

$$\begin{aligned} F(x, y) = (A - a)(B - b) \{ f(x, y) - f(x, b) - f(a, y) + f(a, b) \} \\ - (x - a)(y - b) \{ f(A, B) - f(A, b) - f(a, B) + f(a, b) \}. \end{aligned}$$

Evidently

$$\begin{aligned} F(A, B) = F(A, b) = F(a, B) = F(a, b) \\ = 0, \end{aligned}$$

and therefore there is a point (ξ, η) for which $DF = 0$. But

$$DF = (A - a)(B - b) Df - [A, B; a, b],$$

which proves the theorem.

* To prove the corresponding theorem for one variable it is enough to assume the existence of the differential coefficient without its continuity. I do not at present see how to dispense with the continuity of Df in § 2.

§ 5. If $Df=0$ throughout (a, A, b, B) , f is the sum of a function of x and a function of y .

For if $a < x < A, \quad b < y < B,$

$$\begin{aligned} f(x, y) - f(x, b) - f(a, y) + f(a, b) \\ = (x-a)(y-b) Df(\xi, \eta) \\ (a \leq \xi \leq x, \quad b \leq \eta \leq y) \\ = 0, \end{aligned}$$

$$f(x, y) = f(x, b) + f(a, y) - f(a, b).$$

As Df vanishes whenever f is the sum of two such functions it follows that the necessary and sufficient condition that Df should be everywhere zero is that f should be the sum of a function of x and one of y .

It is interesting to compare the solutions of the equations

$$(1) \quad Df=0,$$

$$(2) \quad \frac{\partial^2 f}{\partial x \partial y} = 0,$$

$$(3) \quad \frac{\partial^2 f}{\partial y \partial x} = 0.$$

Each solution is of the form

$$\phi(x) + \psi(y).$$

In the case of (1) the functions are absolutely arbitrary (in the sense of the theory of functions of a real variable). In the case of (2), ϕ is absolutely arbitrary, while ψ must have a derivate; and in the case of (3) these conditions are reversed.

It is an immediate deduction that the complete solution of

$$Df = \theta(x, y),$$

$\theta(x, y)$ being any continuous function, is

$$f = \int_a^x \int_b^y \theta(x, y) \, dx \, dy + \phi(x) + \psi(y),$$

ϕ, ψ being again quite arbitrary.

END OF VOL. XXXII.

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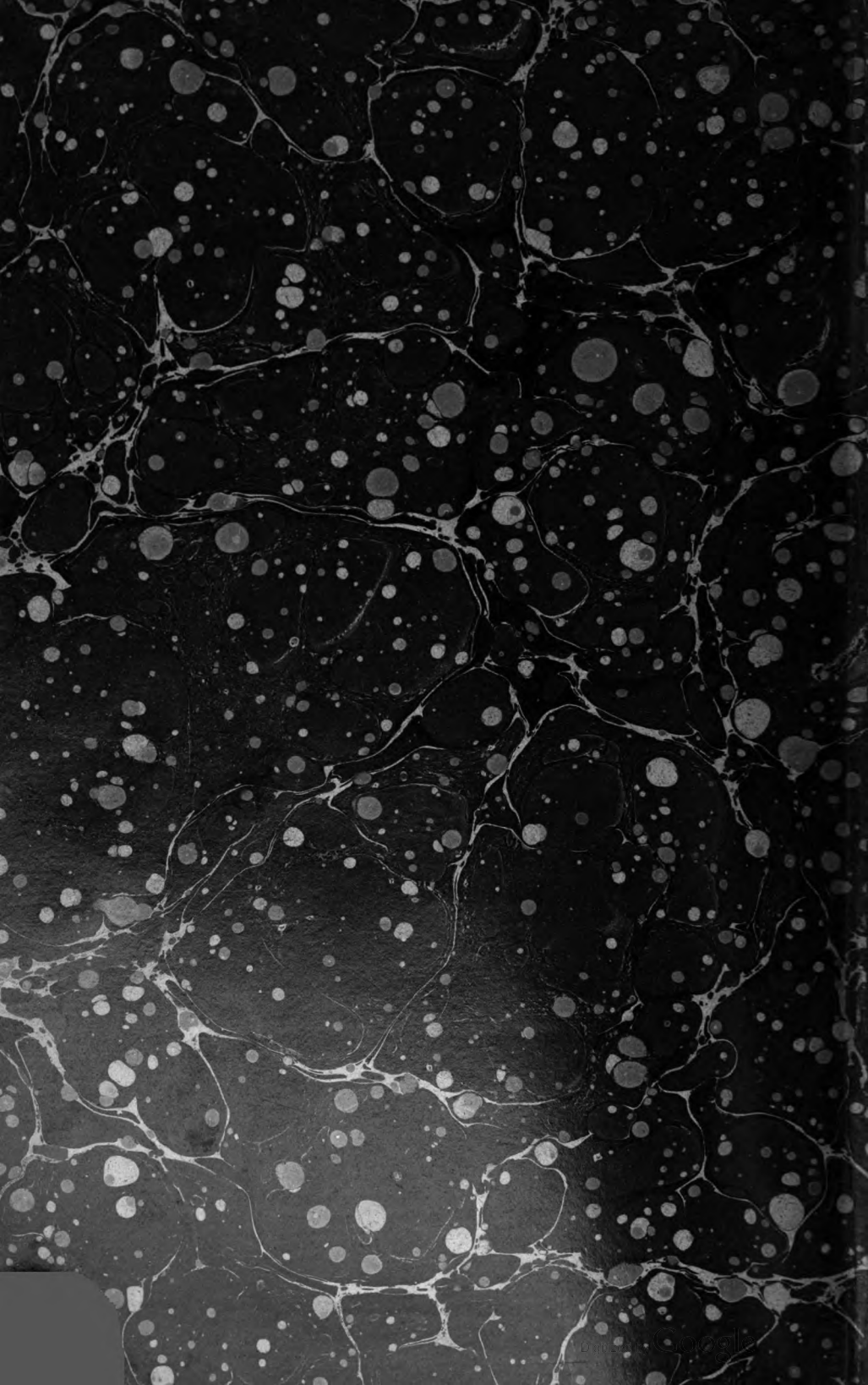
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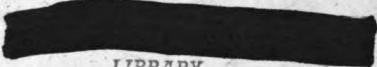
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